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Semiclassical Asymptotics of Eigenvalues for Dirac Operators with Magnetic Fields

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The main purpose of the present paper is to investigate the semiclassical asymptotics of eigenvalues for the Dirac operator with magnetic fields. In the case of the Schrödinger operator with magnetic field, this problem was recently solved by Matsumoto. We show that the nth positive eigenvalue of the Dirac operator behaves like that of the associated Schrödinger operator via unitary equivalence of their spectral measures.

Key Words: Dirac operators; Schrödinger operators; magnetic fields; semiclassical asymptotics; eigenvalues.

1. INTRODUCTION

In this paper we consider semiclassical asymptotics for the Dirac operator with a magnetic vector potential $\mathbf{a} = (a_1, a_2)$ and a scalar potential V defined by

$$\mathbf{H}_{V,\sigma}^{h,c}(\mathbf{a}) = \sum_{j=1}^{2} c\sigma_j \left(-ih\frac{\partial}{\partial x_j} - a_j \right) + c^2\sigma_3 - c^2 + V, \qquad h > 0. \quad (1.1)$$

Here $c \geq 1$ is a constant (the velocity of light) and $\sigma = (\sigma_j)_{j=1,2,3}$ is a system of 2×2 Hermitian symmetric matrices satisfying the relations

$$\begin{cases} \sigma_j\sigma_k + \sigma_k\sigma_j = 2\,\delta_{jk} I, & \text{for } j, k = 1, 2, \\ \sigma_3 = -i\sigma_1\sigma_2, \end{cases} \quad (1.2)$$

where I and δ_{kj} stand for the identity matrix and the Kronecker symbol, respectively. We assume that a_1, a_2, and V are real-valued smooth func-

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tions whose derivatives of all orders are bounded. It is well known that $\mathbf{H}_{V,\sigma}^{h,c}(\mathbf{a})$ becomes essentially self-adjoint on $L^2(\mathbf{R}^2:\mathbf{C}^2)$ (cf. Chernoff [1]).

Let us make assumptions on V with some preparation of notations:

$$\begin{array}{ll} \text{(i)} & 0 \le V(x) < 2 \text{ for } x \in \mathbf{R}. \\ \text{(ii)} & \text{The set of the zeros of } V \text{ is a finite subset of} \\ & \{p_s\}_{s=1}^N \text{ of } \mathbf{R}^2 \text{ and the Hessian } \nabla^2 V(p_s) \text{ is} \\ & \text{non-degenerate for } s = 1, \dots, N. \\ \text{(iii)} & \liminf_{|x|\to\infty} V(x) = E_- > 0 \text{ and } \limsup_{|x|\to\infty} V(x) = E_+ < 2. \end{array} \tag{1.3}$$

Under the assumption (1.3), $\mathbf{H}_{V,\sigma}^{h,c}(\mathbf{a})$ has no essential spectrum on $(-2c^2 + E_+, E_-)$. It is also well known that each positive eigenvalue under the positive essential spectrum has a finite number of multiplicity. We denote them by $\{\lambda_n^c(h)\}$,

$$0 < \lambda_1^c(h) \le \lambda_2^c(h) \le \cdots \le \lambda_n^c(h) \le \cdots,$$

counting with multiplicities. We remark that the Dirac operator is independent of the choice of σ_j satisfying (1.2). Indeed, if $\tilde{\sigma} = (\tilde{\sigma}_j)_{j=1,2,3}$ also satisfies (1.2), there exists a unitary matrix M such that $\sigma_j = M\tilde{\sigma}_j M^*$ for $j = 1,2,3$. In particular we have $\mathbf{H}_{V,\sigma}^{h,c}(\mathbf{a}) = M\mathbf{H}_{V,\tilde{\sigma}}^{h,c}(\mathbf{a})M^*$, namely, the spectrum of $\mathbf{H}_{V,\sigma}^{h,c}(\mathbf{a})$ is equal to that of $\mathbf{H}_{V,\tilde{\sigma}}^{h,c}(\mathbf{a})$. Hence, we take

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

without loss of generality. We also concentrate on the case when $c = 1$ without loss of generality (see [4, Sect. 2.1], for example). Denote $\mathbf{H}_V^h(\mathbf{a}) = \mathbf{H}_{V,\sigma}^{h,1}(\mathbf{a})$ and $\lambda_n(h) = \lambda_n^1(h)$ simply.

Let us introduce the harmonic oscillator $H(k_1, k_2, \beta, S_p)$ on $L^2(\mathbf{R}^2) = L^2(\mathbf{R}^2; \mathbf{C})$, $k_1, k_2 > 0$, $\beta \in \mathbf{R}$, with spin $S_p \in \mathbf{R}$ under a uniform magnetic field on $\mathbf{R}^2$ by

$$H(k_1, k_2, \beta, S_p) = -\frac{1}{2}\sum_{j=1}^{2}\left(\frac{\partial}{\partial x_j} - i\theta_j\right)^2 + S_p\beta + \frac{k_1^2 x_1^2 + k_2^2 x_2^2}{2}, \tag{1.4}$$

where $\theta_1(x) = -\beta x_2/2$ and $\theta_2(x) = \beta x_1/2$. For $s = 1, \dots, N$, we write by $\{\mu_n^s(S_p)\}_{n=1}^\infty$ the eigenvalues of $H(k_1^s, k_2^s, B(p_s), S_p)$ counting with multiplicities, where k_1^s and k_2^s are the eigenvalues of $\nabla^2 V(p_s)$ and $B(x) \equiv \partial a_2(x)/\partial x_1 - \partial a_1(x)/\partial x_2$. Let $\{l_n(S_p)\}_{n=1}^\infty$, $l_1(S_p) \le l_2(S_p) \le \cdots$, be the rearrangement of $\bigcup_{s=1}^N \{\mu_n^s(S_p)\}_{n=1}^\infty$. In this paper we consider the case when $S_p = \pm 1/2$. Hence we write $l_n^+ = l_n(1/2)$ and $l_n^- = l_n(-1/2)$ simply.

Our main result is the following.

THEOREM 1. *Under the assumption* (1.3), *we have*

$$\lim_{h \to 0} \lambda_n(h)/h = l_n^-$$

for each $n \in \mathbf{N}$.

Matsumoto considered the same problem for the Schrödinger operator with magnetic field

$$\mathbf{S}_V^h(\mathbf{a}, S_p) = -\frac{1}{2} \sum_{j=1}^{2} \left(h \frac{\partial}{\partial x_j} - i a_j \right)^2 + S_p B + V \tag{1.5}$$

on $L^2(\mathbf{R}^2)$. Let $\mu_n(h, S_p)$ be the nth eigenvalue (counting with multiplicities) of $\mathbf{S}_V^h(\mathbf{a}, S_p)$. He obtained the following theorem. (See also Proposition 2.2.)

THEOREM 2 (Matsumoto [2]). *Under the assumption* (1.3), *it holds that*

$$\lim_{h \to 0} \mu_n(h, S_p)/h = l_n(S_p)$$

for each $n \in \mathbf{N}$ *and* $S_p \in \mathbf{R}$.

(See [3] for the Schrödinger operators without magnetic fields.) From this result, for Theorem 1, it suffices to show the following theorem.

THEOREM 3. *Under the assumption* (1.3), *we have*

$$\lim_{h \to 0} \lambda_n(h)/h = \lim_{h \to 0} \mu_n(h, -1/2)/h$$

for each $n \in \mathbf{N}$.

For the Dirac operators without magnetic fields, Wang [5] obtained the same result as Theorem 1. On the other hand, it is also known that

$$\lim_{c \to \infty} \left(\lambda_n^c(h) - \mu_n(h, -1/2) \right) = 0.$$

It was obtained through the convergence of the resolvent (see Thaller [4]). We modify their methods to prove Theorem 3.

2. PROOF OF THEOREM 3

Let us define an essentially self-adjoint operator

$$\mathbf{L}_V^h(\mathbf{a}) = -\frac{1}{2}\sum_{j=1}^{2}\left(h\frac{\partial}{\partial x_j} - ia_j\right)^2 + ih\frac{B}{2}\sigma_1\sigma_2 + V \tag{2.1}$$

on $L^2(\mathbf{R}^2 : \mathbf{C}^2)$. Note that

$$\mathbf{L}_V^h(\mathbf{a}) = \begin{pmatrix} \mathbf{S}_V^h(\mathbf{a}, -1/2) & 0 \\ 0 & \mathbf{S}_V^h(\mathbf{a}, 1/2) \end{pmatrix}. \tag{2.2}$$

Hence the spectrum of $\mathbf{L}_V^h(\mathbf{a})$ is obtained as a union of that of $\mathbf{S}_V^h(\mathbf{a}, -1/2)$ and that of $\mathbf{S}_V^h(\mathbf{a}, 1/2)$.

Before proving Theorem 3, we prepare the following theorem by noting that $V - V^2/2$ satisfies the assumptions (1.3) with the same zeros $\{p_s\}_{s=1}^N$ in (ii) as those of V and that $\nabla^2 V(p_s) = \nabla^2(V - V^2/2)(p_s)$, $s = 1, \dots, N$. We denote the inner product, the Hilbertian norm, and the operator norm on $L^2(\mathbf{R}^2)$ or $L^2(\mathbf{R}^2 : \mathbf{C}^2)$ by $(\cdot, \cdot)$, $\|\cdot\|$, and $\|\cdot\|_{op}$, respectively.

THEOREM 2.1. *Let $z \in \mathbf{C} - \{l_n^-, l_n^+\}_{n=1}^\infty$. Then, $(\frac{1}{h}\mathbf{H}_V^h(\mathbf{a}) - z)^{-1}$ and $(\frac{1}{h}\mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z)^{-1}$ exist for all sufficiently small $h > 0$. Furthermore we have*

$$\lim_{h\to 0}\sup_{z\in K}\left\|\left(\frac{1}{h}\mathbf{H}_V^h(\mathbf{a}) - z\right)^{-1} - P\left(\frac{1}{h}\mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z\right)^{-1}\right\|_{op} = 0$$

for every compact subset K of $\mathbf{C} - \{l_n^-, l_n^+\}_{n=1}^\infty$, where $P = (I + \sigma_3)/2$.

The following proposition was obtained by Matsumoto [2]. (See also Theorem 2.)

PROPOSITION 2.2. (i) *Let $z \in \mathbf{C} - \{l_n^-, l_n^+\}_{n=1}^\infty$. Then $(\frac{1}{h}\mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z)^{-1}$ exists for all sufficiently small $h > 0$.*

(ii) *Let K be a compact subset of $\mathbf{C} - \{l_n^-, l_n^+\}_{n=1}^\infty$. If $h > 0$ is small enough, there exists a constant $c_1 = c_1(K, B, V) > 0$ such that*

$$\left\|\left(\frac{1}{h}\mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z\right)^{-1}\right\|_{op} \le c_1$$

for all $z \in K$.

We prove Theorem 2.1 via a series of lemmas. The next lemma is obtained from a simple calculation.

LEMMA 2.3. *Let*

$$A(h) = \sum_{j=1}^{2} \frac{\sigma_j}{2}\left(-ih\frac{\partial}{\partial x_j} - a_j\right) - \frac{V}{2} + h\frac{z}{2}.$$

Then we have

$$\left(\frac{1}{h}\mathbf{H}_V^h(\mathbf{a}) - z\right)(P + A(h)) = \left(\frac{1}{h}\mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z\right) + F(x) - h\frac{z^2}{2},$$

where $F(x) = zV + i\sum_{j=1}^{2}(\sigma_j/2)(\partial V/\partial x_j)$.

LEMMA 2.4. *We have*

$$\lim_{h\to 0}\sup_{z\in K}\left\| F\left(\frac{1}{h}\mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z\right)^{-1}\right\|_{op} = 0$$

for every compact subset K *of* $\mathbf{C} - \{l_n^-, l_n^+\}_{n=1}^{\infty}$.

Proof. Let $\varepsilon > 0$ be an arbitrary constant and K a compact subset of $\mathbf{C} - \{l_n^-, l_n^+\}_{n=1}^{\infty}$. By Proposition 2.2, for small enough $h > 0$, there exists a constant $c_1 = c_1(K, B, V) > 0$ such that $\sup_{z\in K}\|(\frac{1}{h}\mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z)^{-1}\|_{op} \le c_1$. We choose $\phi \in L^2(\mathbf{R}^2, \mathbf{C}^2)$ so that $\|(\frac{1}{h}\mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z)\phi\| = 1$. For $s = 1, \dots, N$, since $F(p_s) = 0$, there exist neighborhoods $O(p_s)$ of p_s such that $|F(x)| < (\varepsilon/\sqrt{2}\,c_1)$ for $x \in O(p_s)$. Then we have

$$\int_{\bigcup_{s=1}^{N} O(p_s)} |F\phi|^2\,dx \le \frac{\varepsilon^2}{2c_1^2}\|\phi\|^2 \le \frac{\varepsilon^2}{2} \tag{2.3}$$

for all $0 < h < \delta$ and $z \in K$. On the other hand,

$$\begin{aligned}
&\left(1 - \frac{N_0}{2}\right)\frac{1}{h}\int_{\mathbf{R}^2 - \bigcup_{s=1}^{N} O(p_s)} \phi V\phi^*\,dx \\
&\quad\le \frac{1}{h}\left(\phi, \left(V - \frac{V^2}{2}\right)\phi\right) \\
&\quad\le \left(\phi, \frac{1}{h}\left[-\frac{1}{2}\sum_{j=1}^{2}\left(h\frac{\partial}{\partial x_j} - ia_j\right)^2 + \left(V - \frac{V^2}{2}\right)\right]\phi\right) \\
&\quad\le \left|\left(\phi, \left(\frac{1}{h}\mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z\right)\phi\right)\right| + \frac{1}{2}|(\phi, B\phi)| + |(\phi, z\phi)| \\
&\quad\le c_1\left\{1 + \left(\frac{1}{2}L_0 + |z|\right)c_1\right\},
\end{aligned} \tag{2.4}$$

where $N_0 \equiv \sup_{x \in \mathbf{R}^2} V(x) < 2$ and $L_0 \equiv \sup_{x \in \mathbf{R}^2} |B(x)|$. Hence, we obtain

$$\begin{aligned}\int_{\mathbf{R}^2 - \bigcup_{s=1}^N O(p_s)} |F\phi|^2\, dx &\leq F_0^2 \int_{\mathbf{R}^2 - \bigcup_{s=1}^N O(p_s)} |\phi|^2\, dx \\ &\leq F_0^2 \frac{1}{M_0} \int_{\mathbf{R}^2 - \bigcup_{s=1}^N O(p_s)} \phi V \phi^*\, dx \\ &\leq hF_0^2 \frac{2c_1}{M_0(2 - N_0)} \left\{ 1 + \left(\frac{1}{2} L_0 + |z| \right) c_1 \right\}, \end{aligned} \tag{2.5}$$

where $F_0 \equiv \sup_{x \in \mathbf{R}^2 - \bigcup O(p_s)} |F(x)|$ and $M_0 \equiv \inf_{x \in \mathbf{R}^2 - \bigcup O(p_s)} V(x) > 0$. This implies the existence of $\delta > 0$ such that

$$\int_{\mathbf{R}^2 - \bigcup_{s=1}^N O(p_s)} |F\phi|^2\, dx \leq \frac{\varepsilon^2}{2} \tag{2.6}$$

for all $0 < h < \delta$ and $z \in K$. Combining (2.3) with this, we conclude that

$$\|F\phi\|^2 = \int_{\bigcup_{s=1}^N O(p_s)} |F\phi|^2\, dx + \int_{\mathbf{R}^2 - \bigcup_{s=1}^N O(p_s)} |F\phi|^2\, dx \leq \varepsilon^2, \tag{2.7}$$

which completes the proof. ∎

The next lemma is a straight consequence of Proposition 2.2 and Lemma 2.4.

LEMMA 2.5. *For* $z \in \mathbf{C} - \{l_n^-, l_n^+\}_{n=1}^\infty$, $\{I + (F - h\frac{z^2}{2})(\frac{1}{h}\mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z)^{-1}\}^{-1}$ *exists for all sufficiently small* $h > 0$. *Furthermore we have*

$$\lim_{h \to 0} \sup_{z \in K} \left\| \left\{ I + \left(F - h\frac{z^2}{2} \right) \left(\frac{1}{h} \mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z \right)^{-1} \right\}^{-1} - I \right\|_{op} = 0$$

for every compact subset K of $\mathbf{C} - \{l_n^-, l_n^+\}_{n=1}^\infty$.

LEMMA 2.6. *We have*

$$\lim_{h \to 0} \sup_{z \in K} \left\| A(h) \left(\frac{1}{h} \mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z \right)^{-1} \right\|_{op} = 0$$

for every compact subset K of $\mathbf{C} - \{l_n^-, l_n^+\}_{n=1}^\infty$.

Proof. As a similar method of the proof of Lemma 2.4, we can prove that $\lim_{h \to 0} \sup_{z \in K} \|(-\frac{V}{2} + h\frac{z}{2})(\frac{1}{h}\mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z)^{-1}\|_{op} = 0$. It suffices to prove that

$$\lim_{h \to 0} \sup_{z \in K} \left\| \left(\sum_{j=1}^2 \frac{\sigma_j}{2} \left(-ih\frac{\partial}{\partial x_j} - a_j \right) \right) \left(\frac{1}{h} \mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z \right)^{-1} \right\|_{op} = 0. \tag{2.8}$$

For $\phi \in L^2(\mathbf{R}^2 : \mathbf{C}^2)$ satisfying $\|(\frac{1}{h}\mathbf{L}^h_{V-V^2/2}(\mathbf{a}) - z)\phi\| = 1$, we have

$$\begin{aligned}
&\left\| \sum_{j=1}^{2} \frac{\sigma_j}{2}\left(-ih\frac{\partial}{\partial x_j} - a_j\right)\phi \right\|^2 \\
&\quad = \left(\phi, \left\{\sum_{j=1}^{2} \frac{\sigma_j}{2}\left(-ih\frac{\partial}{\partial x_j} - a_j\right)\right\}^2 \phi\right) \\
&\quad = \frac{1}{2}\left(\phi, \left\{h\left(\frac{1}{h}\mathbf{L}^h_{V-V^2/2}(\mathbf{a}) - z\right) + hz - V + \frac{V^2}{2}\right\}\phi\right) \\
&\quad \leq \frac{c_1}{2}\left[h(1 + c_1|z|) + \left\|\left(V - \frac{V^2}{2}\right)\phi\right\|\right]. \qquad (2.9)
\end{aligned}$$

By using the same method as in the proof of Lemma 2.4, we can show that the right hand side of (2.9) tends to zero uniformly on $z \in K$ as $h \to 0$. ∎

Proof of Theorem 2.1. By Lemmas 2.3 and 2.5, it holds that

$$\begin{aligned}
\left(\frac{1}{h}\mathbf{H}^h_V(\mathbf{a}) - z\right)^{-1} &= (P + A(h))\left(\frac{1}{h}\mathbf{L}^h_{V-V^2/2}(\mathbf{a}) - z\right)^{-1} \\
&\quad \times \left\{1 + \left(F - h\frac{z^2}{2}\right)\left(\frac{1}{h}\mathbf{L}^h_{V-V^2/2}(\mathbf{a}) - z\right)^{-1}\right\}^{-1}.
\end{aligned} \qquad (2.10)$$

Hence, together with Lemma 2.6, we complete the proof of Theorem 2.1. ∎

Proof of Theorem 3. We choose $M > 0$ so that $M \notin \{l_n^-, l_n^+\}_{n=1}^{\infty}$ and suppose that $l_a^- < M < l_{a+1}^-$ and $l_b^+ < M < l_{b+1}^+$. Namely, we take $a = \min\{n \geq 0;\ M < l_{n+1}^-\}$ and $b = \min\{n \geq 0;\ M < l_{n+1}^+\}$. Set

$$d_M = \min\{|\tau_1 - \tau_2|;\ \tau_1, \tau_2 \in \{0, M, l_1^-, \ldots, l_a^-, l_1^+, \ldots, l_b^+\}, \tau_1 \neq \tau_2\}. \qquad (2.11)$$

From Theorem 2, for all $0 < \varepsilon < d_M/2$, there exists $\delta > 0$ such that if $h < \delta$, then

$$\begin{aligned}
&|\mu_j(h, -1/2)/h - l_j^-| < \varepsilon && \text{if } 1 \leq j \leq a, \\
&\mu_j(h, -1/2)/h > M && \text{otherwise}.
\end{aligned}$$

Let $\{w_j\}_{j=1}^{2a+2}$, $w_1 \le w_2 \le \cdots$, be the rearrangement of 0, M, $\{l_j^- - \varepsilon\}_{j=1}^a$, and $\{l_j^- + \varepsilon\}_{j=1}^a$. We denote by $F_h(\mu)$ and $E_h(\mu)$ the spectral measures associated with $\mathbf{L}_{V-V^2/2}^h(\mathbf{a})$ and $\mathbf{H}_V^h(\mathbf{a})$, respectively. For $j = 1, \dots, 2a+1$ and sufficiently small $h > 0$, we define operators

$$P_j(h) = -\frac{1}{2\pi i}\oint_{\Gamma_j} P\left(\frac{1}{h}\mathbf{L}_{V-V^2/2}^h(\mathbf{a}) - z\right)^{-1} dz = F_h(w_{j+1}) - F_h(w_j),$$

$$P_j^0(h) = -\frac{1}{2\pi i}\oint_{\Gamma_j}\left(\frac{1}{h}\mathbf{H}_V^h(\mathbf{a}) - z\right)^{-1} dz = E_h(w_{j+1}) - E_h(w_j),$$

where Γ_j is a positively oriented circle with the center $(w_j + w_{j+1})/2$ and the radius $(w_{j+1} - w_j)/2 > 0$. By Theorem 2.1 we notice that $\|P_j(h) - P_j^0(h)\|_{op} < 1$ for all sufficiently small $h > 0$. Then we define the operator

$$U_j(h) \equiv \left(1 - \left\{P_j(h) - P_j^0(h)\right\}^2\right)^{-1/2} \times\left\{P_j(h)P_j^0(h) + \left(1 - P_j(h)\right)\left(1 - P_j^0(h)\right)\right\},$$

which is unitary and satisfies $P_j(h)U_j(h) = U_j(h)P_j^0(h)$. Namely,

$$\left(F_h(w_{j+1}) - F_h(w_j)\right)U_j(h) = U_j(h)\left(E_h(w_{j+1}) - E_h(w_j)\right).$$

Therefore, we have obtained that

$$\lim_{h\to 0}\mu_n(h, -1/2)/h = \lim_{h\to 0}\lambda_n(h)/h$$

for $n = 1, \dots, a$, which completes the proof of Theorem 3. ∎

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